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ON REGULARITY, AMENABILITY AND OPTIMAL FACTORING STRATEGIES FOR RELIABILITY COMPUTATIONS

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Abstract

In the present paper we investigate three classes of reliability systems: regular clutters, total amenable clutters and clutters where a certain factoring strategy for computing reliability is in a sense optimal. In the first part of the paper it is shown that the class of regular clutters is strictly contained in the class of total amenable clutters, while the class of total amenable clutters is strictly contained in the third class. In the second part of the paper we present some important new properties of regular clutters. Especially, we provide two new types of regular clutters related to undirected networks.

RELIABILITY COMPUTATIONS; REGULARITY; DOMINATION; AMENABILITY; UNDIRECTED NETWORKS; MATROIDS

1. Introduction.

The study of optimal factoring strategies for reliability computations originates from the work of Satyanarayana and Chang [8] and Chang [3]. They studied such strategies in the context of multiterminal undirected network systems, and discovered that there is a close relation between such strategies and a certain invariant called the domination. This relation derives from the fact that the domination of such network systems possesses a certain recursiveness property. In Barlow and Iyer [2] a reliability system where the domination possesses this recursiveness property, is called totally amenable. As pointed out both in Huseby [4] and in Barlow and Iyer [2], the factoring strategy derived by Satyanarayana and Chang is optimal for all totally amenable reliability systems. Huseby [5] provides a sufficient condition for a reliability system to be totally amenable. A system satisfying this condition, is called regular.

It has been speculated on (see Huseby [4] and Barlow and Iyer [2]) whether or not regularity is also a necessary condition for total amenability. In the first part of this paper we shall investigate this problem in the light of the paper by Seymour [9]. The significance of this paper in relation to the above problem was first observed by Majumdar [7], but the conclusions we draw here, are a bit different. Specifically, it turns out that regularity and total amenability are not equivalent concepts. As we shall see, the results of Seymour [9] may also be used to show that total amenability is not a necessary condition for the factoring strategy of Satyanarayana and Chang to be optimal.

2. Basic concepts and results.

A *clutter* is an ordered pair (E, $\mathscr{P}$), where E is a finite set, and $\mathscr{P}$ is a family of incomparable subsets of E, (i.e. no set is a proper subset of another). It may happen that some component in E does not appear in any of the sets in $\mathscr{P}$. Such a component is called *irrelevant*. Conversely, a component contained at least one of the sets $\mathscr{P}$, is called *relevant.* A clutter with only relevant components is said to be *coherent*. We also introduce the *structure function* of a clutter, (E,$\mathscr{P}$), usually denoted by ϕ, defined for all $A\subseteq E$, such that $\phi(A)$ is one if $P\subseteq A$ for some $P\in \mathscr{P}$, and zero otherwise. We observe that ϕ is binary and nondecreasing (i.e. $A\subseteq B$ implies $\phi(A) \leq \phi(B)$). Conversely, if ϕ is a binary nondecreasing function defined for all $A\subseteq E$, then ϕ is a structure function of a clutter (E, $\mathscr{P}$) where $\mathscr{P}$ is the family of minimal subsets $P\subseteq E$ such that $\phi(P) = 1$. Thus, especially (E, $\mathscr{P}$) is uniquely determined by its structure function.

If e is an element of a clutter (E, $\mathscr{P}$), then $\mathscr{P}_{+e}$ denotes the family of minimal sets of the form (P\e) where $P\in\mathscr{P}$, and $\mathscr{P}_{-e}$ denotes the family of all sets in $\mathscr{P}$ which do not contain e. The clutters (E\e, $\mathscr{P}_{+e}$) and (E\e, $\mathscr{P}_{-e}$) are called respectively the *contraction* and *restriction* of (E, $\mathscr{P}$) with respect to e. (F,$\mathscr{Q}$) is called a *minor* of (E, $\mathscr{P}$) if it can be obtained from (E, $\mathscr{P}$) by performing a sequence of contractions and restrictions. It is easily seen that if e is irrelevant, then $\mathscr{P}_{+e} = \mathscr{P}_{-e} = \mathscr{P}$. Thus, both the contraction and restriction of (E, $\mathscr{P}$) are obtained by simply deleting e from the clutter set. Conversely, this also implies that if (F, $\mathscr{Q}$) is obtained from (E, $\mathscr{P}$) by deleting irrelevant components, then (F, $\mathscr{Q}$) is a minor of (E, $\mathscr{P}$).

A *cut set* of a clutter (E, $\mathscr{P}$) is a set $C\subseteq E$ such that $C\cap P\neq\emptyset$ for all $P\in\mathscr{P}$. If $\mathscr{C}$ is the family of minimal cut sets of (E, $\mathscr{P}$), we call (E, $\mathscr{C}$) the *dual* clutter of (E, $\mathscr{P}$).

For all families of sets F and elements f, we denote by F_f the family of sets $F\in F$ such that $f\in F$. If (E,$\mathscr{P}$) is a clutter with dual (E,$\mathscr{C}$), we say that e, $f\in E$ ($e\neq f$) are in *series* (*parallel*) if $\mathscr{P}_e = \mathscr{P}_f$ ($\mathscr{C}_e = \mathscr{C}_f$). The concepts of series and parallel elements in clutters correspond of course to the usual notions of series and parallel components (or edges) in reliability and graph theory. It is usual to simplify clutters by replacing series and parallel components by a single component. We refer to such operations as *series* and *parallel reductions*, or more briefly, *s-p-reductions*. If (E,$\mathscr{P}$) is a clutter, we denote by (E^r,$\mathscr{P}^r$) the clutter obtained from (E,$\mathscr{P}$) by performing all possible s-p-reductions and deleting all irrelevant elements. It is easy to see that for a clutter a series (parallel) reduction is isomorphic to a contraction (restriction). Hence, (E^r,$\mathscr{P}^r$) is a minor of (E,$\mathscr{P}$). If $|E^r| = 0$, (i.e. the clutter contains no relevant elements) (E,$\mathscr{P}$) is called *trivial*, while if $|E^r| = 1$, (E,$\mathscr{P}$) is called an *s-p-system*. Finally, if $|E^r| = |E| > 1$, (E,$\mathscr{P}$) is said to be *s-p-complex*.

A reliability system is an ordered triple (E,$\mathscr{P}$,π), where E is a finite set of elements, called components, being either functioning or failed, $\mathscr{P}$ is a family of incomparable subsets of E, interpreted as the *minimal path sets* of the system, and π is a probability measure on the set of component states. In this paper we only consider systems of *statistically independent* components. For such systems, the probability measure is uniquely determined by the vector of component reliabilities (i.e., the functioning probabilities of the components. If **p** is such a vector, we use the notation $(E,\mathscr{P},\mathbf{p})$ for the reliability system. The *reliability* of $(E,\mathscr{P},\mathbf{p})$, i.e. the probability that at least one minimal path set is functioning, is denoted by $h(\mathscr{P},\mathbf{p})$.

If $e \in E$, and the reliability of e is p_e, the following factoring formula for computing $h(\mathscr{P},\mathbf{p})$ is well-known:

$$(2.1.) \qquad h(\mathscr{P},\mathbf{p}) = p_e\, h(\mathscr{P}_{+e},\mathbf{p}^{E\setminus e}) + (1 - p_e)\, h(\mathscr{P}_{-e},\mathbf{p}^{E\setminus e})$$

where $\mathbf{p}^{E\setminus e}$ denotes the subvector of $\mathbf{p}$ corresponding to the set $(E\setminus e)$. The element e is referred to as the *factoring element*. Thus, computing the reliability of $(E,\mathscr{P},\mathbf{p})$ is reduced to computing the reliability of two simpler systems, corresponding to the contraction and restriction of the clutter.

By updating the component reliabilities in a certain way, it is possible to ensure that the reliability of the system is preserved under s-p-reductions. Furthermore, since irrelevant components do not contribute to the system reliability, they can always be deleted from the system without altering the reliability. Hence, if $(E,\mathscr{P},\mathbf{p})$ is a reliability system, then there exists a vector $\mathbf{q}$ such that the reliabilities of $(E,\mathscr{P},\mathbf{p})$ and $(E^r,\mathscr{P}^r,\mathbf{q})$ are equal, and $h(\mathscr{P},\mathbf{p})$ may be computed by:

$$(2.2.) \qquad h(\mathscr{P},\mathbf{p}) = h(\mathscr{P}^r,\mathbf{q})$$

By combining (2.1.) and (2.2.) a recursive algorithm for computing $h(\mathscr{P},\mathbf{p})$, known as the *factoring algorithm*, is constructed. For more details about this algorithm, we refer to Satyanarayana and Chang [8] or Huseby [4].

The efficiency of this algorithm depends strongly on the number of times the formula (2.1.) is used in the computations. It is easily seen that this number in general is dependent on the choice of the factoring element e. In Satyanarayana and Chang [8] and Chang [3] it is shown that for certain types of reliability systems, related to undirected networks, an optimal strategy (in the sense that the number of factoring operations is minimized) is obtained by choosing e such that both minors are coherent. Huseby [4] shows that for any coherent, s-p-complex clutter, there always exists such a factoring element. At every step of the recursion the factoring algorithm starts out by deleting irrelevant components and performing all possible s-p-reductions. Hence, whenever a factoring operation is needed (i.e., (2.1.) has to be applied), the system to be decomposed is always coherent and s-p-complex, and so by Huseby [4] the strategy of Satyanarayana and Chang may be applied.

More recent studies of this problem have shown that this factoring strategy in fact is optimal under far more general conditions. (See Barlow and Iyer [2], Huseby [4] and Huseby [5].) In general we shall say that a clutter is *c-optimal* if the clutter is s-p-complex, coherent and the factoring strategy of Satyanarayana and Chang is optimal in the above sense. Moreover, a clutter is *totally c-optimal* if every s-p-complex, coherent minor of the clutter, as well as the clutter itself if it is s-p-complex and coherent, is c-optimal. We denote by $\boldsymbol{C}$ the class of all totally c-optimal clutters.

It is well-known that the structure function, ϕ, of a clutter $(E,\mathscr{P})$ may be written on the form:

(2.3) $$\phi(A) = \sum_{B \subseteq A} \delta(B)$$

where δ is a (unique) integer-valued function, called the *signed domination function* of the clutter. (See e.g. Huseby [4].) The *domination* of the clutter is defined as $D(\mathscr{P}) = |\delta(E)|$. $(E,\mathscr{P})$ is said to be *amenable* if the following relation holds true for all relevant $e \in E$:

(2.4.) $$D(\mathscr{P}) = D(\mathscr{P}_{+e}) + D(\mathscr{P}_{-e})$$

If $(E,\mathscr{P})$ and all its minors are amenable, we say that $(E,\mathscr{P})$ is *totally amenable.* We denote by $\boldsymbol{A}$ the class of all totally amenable clutters. Note that by definition both $\boldsymbol{C}$ and $\boldsymbol{A}$ are closed under minor operations. Barlow and Iyer [2] have established the following relation between $\boldsymbol{A}$ and $\boldsymbol{C}$:

(2.5.) $$\boldsymbol{A} \subseteq \boldsymbol{C}$$

A *matroid* is a clutter $(F,\mathscr{M})$ where F is non-empty, $\emptyset \notin \mathscr{M}$, and for all $A, B \in \mathscr{M}$ $(A \neq B)$ and elements $e \in (A \cap B)$, there exists $C \in \mathscr{M}$ such that $C \subseteq (A \cup B) \setminus e$. We say that a clutter $(E,\mathscr{P})$ is *regular* if there exists a matroid $(E \cup x,\mathscr{M})$ where $x \notin E$, such that:

(2.6.) $$(E,\mathscr{P}) = (E,(\mathscr{M}_x)_{+x}).$$

If so, we say that $(E \cup x,\mathscr{M})$ is a *corresponding matroid* of $(E,\mathscr{P})$, and x is called the *extension component*. We also write $(E,\mathscr{P}) \leftrightarrow (E \cup x,\mathscr{M})$.

The class of all regular clutters is denoted by $\boldsymbol{R}$. In Huseby [5] it is shown that also $\boldsymbol{R}$ is closed under minor operations. Moreover, it is shown that:

(2.7.) $$\boldsymbol{R} \subseteq \boldsymbol{A}$$

In the next section we shall see that the inclusions (2.5.) and (2.7.) are strict.

3. A characterization of the class of regular clutters.

The concept of regular clutters is in fact identical to the concept of *ports* introduced in Lehman [6]. In Seymour [9] several characterizations of ports are provided. Most important is a socalled *forbidden minor characterization*. Translated into our terminology and notation this result can be stated as follows:

Theorem 3.1. A clutter is regular if and only if it has no minors of the following four types:

(i) $(E^1,\mathscr{P}^1)$ where $E^1 = \{1,2,3,4\}$ and $\mathscr{P}^1 = \{\{1,2\}, \{2,3\}, \{3,4\}\}$

(ii) $(E^2,\mathscr{P}^2)$ where $E^2 = \{1,2,3,4\}$ and $\mathscr{P}^2 = \{\{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}\}$

(iii) $(E^3,\mathscr{P}^3)$ where $E^3 = \{1,2,3,4\}$ and $\mathscr{P}^3 = \{\{1,2\}, \{1,3\}, \{2,3,4\}\}$

(iv) $(E^4,\mathscr{P}^4)$ where $E^4 = \{0,1,\ldots,s\}$ and $\mathscr{P}^4 = \{\{1,2,\ldots,s\}, \{0,1\},\ldots,\{0,s\}\}$, where $s \geq 3$

In Figure 3.1 the logic diagrams of the four forbidden minors are given.

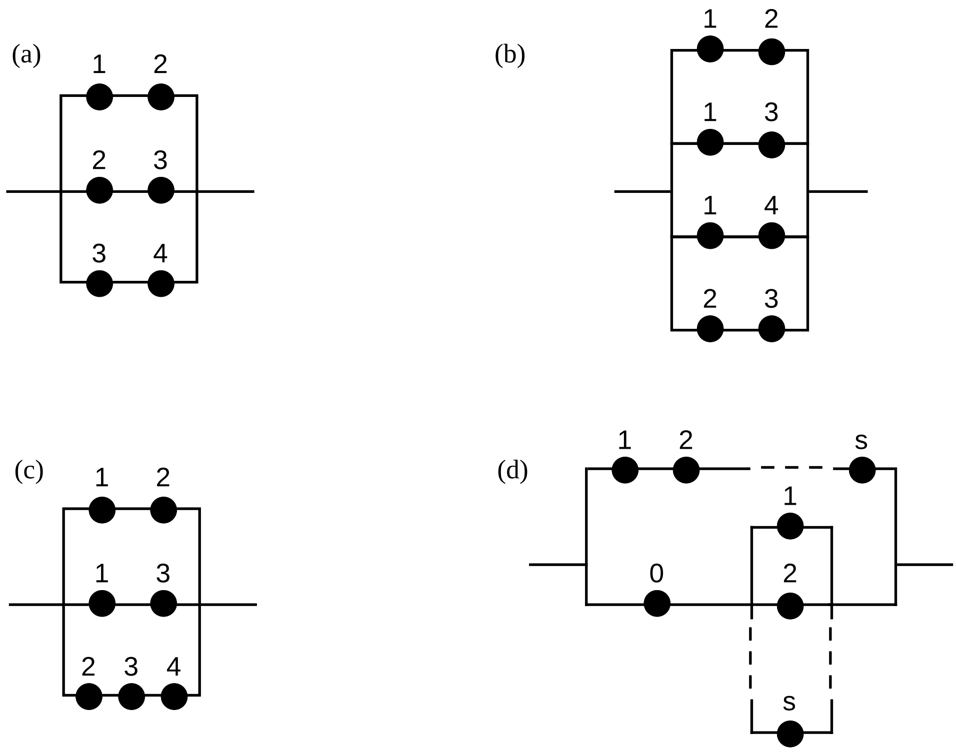

Figure 3.1. The forbidden minors of $\boldsymbol{R}$: (a) $(\mathrm{E}^1,\mathscr{P}^1)$, (b) $(\mathrm{E}^2,\mathscr{P}^2)$, (c) $(\mathrm{E}^3,\mathscr{P}^3)$, (d) $(\mathrm{E}^3,\mathscr{P}^3)$.

Now, if $\boldsymbol{M}$ is a class of clutters, we denote by $\boldsymbol{M}^c$, the class of clutters with no minors in $\boldsymbol{M}$. Obviously, $\boldsymbol{M}^c$ is closed under minor operations. Indeed, $\boldsymbol{M}^c$ is the largest class of clutters closed under minor operations, being disjoint from $\boldsymbol{M}$. That is, if $\boldsymbol{L}$ is another class of clutters closed under minor operations, and such that $\boldsymbol{L}\cap\boldsymbol{M} = \emptyset$, then $\boldsymbol{L}\subseteq\boldsymbol{M}^c$. [To prove this, assume conversely that there exists a clutter $(\mathrm{E},\mathscr{P})\in\boldsymbol{L}$ such that $(\mathrm{E},\mathscr{P})\notin\boldsymbol{M}^c$. Since $\boldsymbol{L}\cap\boldsymbol{M} = \emptyset$, we must have that $(\mathrm{E},\mathscr{P})\notin\boldsymbol{M}$. Furthermore, since $(\mathrm{E},\mathscr{P})\notin\boldsymbol{M}^c$, there must exist a minor $(\mathrm{F},\mathscr{Q})$ of $(\mathrm{E},\mathscr{P})$ such that $(\mathrm{F},\mathscr{Q})\in\boldsymbol{M}$. However, $\boldsymbol{L}$ is assumed to be closed under minor operations, so $(\mathrm{F},\mathscr{Q})\in\boldsymbol{L}$ as well, contradicting the assumption that $\boldsymbol{L}\cap\boldsymbol{M} = \emptyset$.]

More specifically, let $\boldsymbol{M}$ be the class of clutters of the four types described in Theorem 3.1. Then clearly Seymour's result can be stated as: $\boldsymbol{R} = \boldsymbol{M}^c$. If we could show that $\boldsymbol{A}\cap\boldsymbol{M} = \emptyset$, then by the above argument, we would get that $\boldsymbol{A}\subseteq\boldsymbol{R}$. Hence, by (2.7.), we could conclude that $\boldsymbol{A} = \boldsymbol{R}$. Similarly, if we could show that $\boldsymbol{C}\cap\boldsymbol{M} = \emptyset$, then $\boldsymbol{C}\subseteq\boldsymbol{R}$, and thus by (2.5.) and (2.7.) $\boldsymbol{C} = \boldsymbol{R}$. However, as we imediately shall see, none of the above conclusions holds true. To see this, we simply investigate the four types of clutters described in Theorem 3.1. We note first that in order to check total amenability and total c-optimality, it is sufficient to check amenability and c-optimality, since using Theorem 3.1 one can show that any minor of these clutters are regular, and hence also amenable and c-optimal. The proofs are indeed very simple, but in order to explore the different possible cases, we feel that it is illustrating to review them briefly. The results are as follows:

Lemma 3.2. $(\mathrm{E}^1, \mathscr{P}^1) \notin \boldsymbol{A}$, but $(\mathrm{E}^1, \mathscr{P}^1) \in \boldsymbol{C}$.

Proof: It is easily seen that $\mathrm{D}(\mathscr{P}^1) = 0$, while e.g., $\mathrm{D}(\mathscr{P}^1_{+1}) = \mathrm{D}(\mathscr{P}^1_{-1}) = 1$. Hence, $\mathrm{D}(\mathscr{P}^1) \neq \mathrm{D}(\mathscr{P}^1_{+1}) + \mathrm{D}(\mathscr{P}^1_{-1})$, and so $(\mathrm{E}^1,\mathscr{P}^1) \notin \boldsymbol{A}$.

By considering the contractions and restrictions of $(E^1, \mathscr{P}^1)$ w.r.t. the different components, one can verify that $(E^1\backslash 1, \mathscr{P}^1_{+1})$, $(E^1\backslash 1, \mathscr{P}^1_{-1})$, $(E^1\backslash 4, \mathscr{P}^1_{+4})$ and $(E^1\backslash 4, \mathscr{P}^1_{-4})$ are coherent, while $(E^1\backslash 2, \mathscr{P}^1_{+2})$, $(E^1\backslash 2, \mathscr{P}^1_{-2})$, $(E^1\backslash 3, \mathscr{P}^1_{+3})$ and $(E^1\backslash 3, \mathscr{P}^1_{-3})$ are non-coherent. Hence, according to the factoring strategy of Satyanarayana and Chang, one should choose either 1 or 4 as the factoring element. In this case, however, it turns out that regardless of the choice of factoring element, the contraction and restriction are both s-p-systems. Thus, any factoring, especially that of Sayanarayana and Chang is optimal, and so $(E^1, \mathscr{P}^1) \in \boldsymbol{C}$. ■

Lemma 3.3. $(E^2, \mathscr{P}^2) \notin \boldsymbol{A}$, and $(E^2, \mathscr{P}^2) \notin \boldsymbol{C}$.

Proof: Straightforward calculations yield that $D(\mathscr{P}^2) = 1$, while $D(\mathscr{P}^2_{+4}) = 1$ and $D(\mathscr{P}^2_{-4}) = 2$. Hence, $D(\mathscr{P}^2) \neq D(\mathscr{P}^2_{+4}) + D(\mathscr{P}^2_{-4})$, and so $(E^2, \mathscr{P}^2) \notin \boldsymbol{A}$.

Considering the contractions and restrictions of $(E^2, \mathscr{P}^2)$, one can verify that the only component yielding coherent minors is 4. However, while factoring on any of the other components yields s-p-systems as minors, the restriction of $(E^2, \mathscr{P}^2)$ w.r.t. 4 is an s-p-complex clutter. Hence, although the contraction of $(E^2, \mathscr{P}^2)$ is an s-p-system, choosing 4 as the factoring element is clearly not optimal. Hence, $(E^2, \mathscr{P}^2) \notin \boldsymbol{C}$. ■

Lemma 3.4. $(E^3, \mathscr{P}^3) \notin \boldsymbol{A}$, and $(E^3, \mathscr{P}^3) \notin \boldsymbol{C}$.

Proof: It is easy to see that $(E^3, \mathscr{P}^3)$ is the dual of $(E^2, \mathscr{P}^2)$, Due to this fact, the proof of this result is completely similar to that of Lemma 3.3. We skip the details here. ■

Lemma 3.5. $(E^4, \mathscr{P}^4) \in \boldsymbol{A}$ if s≥3 is even, while $(E^4, \mathscr{P}^4) \notin \boldsymbol{A}$ if s≥3 is odd. $(E^4, \mathscr{P}^4) \in \boldsymbol{C}$ for all s≥3.

Proof: Again it is easy to see that $D(\mathscr{P}^4) = 2$ for s≥3 even, and $D(\mathscr{P}^4) = 0$ for s≥3 odd. Moreover, $D(\mathscr{P}^4_{+e}) = D(\mathscr{P}^4_{-4}) = 1$ for all $e \in E^4$. Thus, $(E^4, \mathscr{P}^4) \in \boldsymbol{A}$ for s≥3 even, and $(E^4, \mathscr{P}^4) \notin \boldsymbol{A}$ for s≥3 odd.

In this case all contractions and restrictions of the clutter are coherent s-p-systems. Hence, according to the factoring strategy of Satyanarayana and Chang, any component may be chosen as factoring element. Since also every component is optimal, it is clear that $(E^4, \mathscr{P}^4) \in \boldsymbol{C}$.■

By the above lemmas the following main result of this section is imediate.

Theorem 3.6. The inclusions (2.5.) and (2.7.) are strict.

Proof: By Theorem 3.1 and Lemma 3.5 $(E^4, \mathscr{P}^4)$ is non-regular and totally amenable for s≥3 even. Thus, (2.7.) is strict. Moreover, by Lemma 3.2 $(E^1, \mathscr{P}^1)$ is non-amenable and c-optimal. Thus, (2.5.) is strict. ■

As corollaries we also obtain the following:

Corollary 3.7. A necessary condition for a clutter to be totally amenable is that it has no minors of the types $(E^1, \mathscr{P}^1)$, $(E^2, \mathscr{P}^2)$, $(E^3, \mathscr{P}^3)$, or $(E^4, \mathscr{P}^4)$ for s≥3 odd.

Corollary 3.8. A necessary condition for a clutter to be totally c-optimal is that it has no minors of the types $(E^2, \mathscr{P}^2)$ or $(E^3, \mathscr{P}^3)$.

An interesting question now is whether the conditions given in Corollary 3.7 and Corollary 3.8 is also sufficient conditions for respectively totally amenability and total c-optimality. Especially, one might conjecture that $(E^2, \mathscr{P}^2)$ or $(E^3, \mathscr{P}^3)$ are the only forbidden minors of the class ***C***. We close this section by providing an example showing that at least this is not true.

Example 3.9. Consider the clutter $(E, \mathscr{P})$ where E = {1,2,…,6} and $\mathscr{P}$ = {{1,2}, {2,3}, {3,4,5}, {3,4,6}, {3,5,6}}. A diagram of this clutter is shown in Figure 3.2.

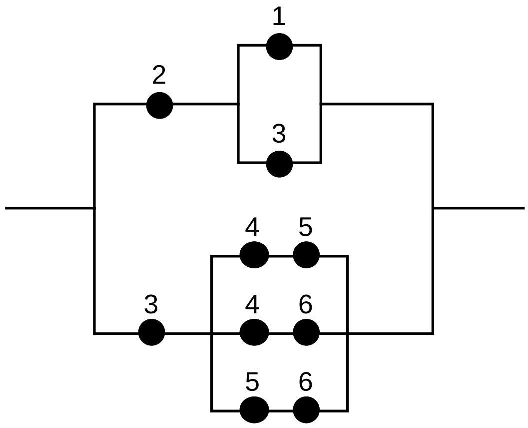

Figure 3.2. A minimal non-c-optimal clutter without minors of the type $(E^2, \mathscr{P}^2)$ or $(E^3, \mathscr{P}^3)$.

By investigating all four-element minors, it can be verified that none of them is of type $(E^2, \mathscr{P}^2)$ or $(E^3, \mathscr{P}^3)$. In fact, all four-element minors are either regular or isomorphic to $(E^1, \mathscr{P}^1)$. Thus, in the light of Corollary 3.8 one might hope that $(E,\mathscr{P})$ is totally c-optimal.

By considering all restrictions and contractions of $(E,\mathscr{P})$, it is seen that coherent minors are obtained by factoring on components 1, 4, 5 or 6, while factoring on 2 or 3 yields non-coherent minors. Thus, if $(E,\mathscr{P})$ is c-optimal, 1, 4, 5 or 6 should be chosen as factoring element. If one of these components is chosen, both minors need to be factored once more in order to obtain the system reliability. If 2 or 3 is chosen, however, only one of the minors needs an additional factoring. Hence, the optimal factoring element is either 2 or 3. Thus, $(E,\mathscr{P})$ is not c-optimal.

It is easy to see that $(E,\mathscr{P})$ is minimal with this property, in the sense that every minor of $(E,\mathscr{P})$ is totally c-optimal. We observe also that $(E,\mathscr{P})$ is obtained from $(E^1, \mathscr{P}^1)$ by replacing the single component 4 by a 2-out-of-3 clutter, consisting of the components 4, 5 and 6. Similar examples can be constructed by replacing component 4 (or 1) by other s-p-complex clutters. Thus, the class of totally c-optimal clutters appear to have a large class of different minimal forbidden minors. ■

4. Properties of regular clutters.

In the previous section we saw that the class ***C*** was richer than both ***A*** and ***R***. However, we still need some efficient way of checking whether a given clutter is totally c-optimal or not. Instead of working on the problem of finding a general condition, we shall explore some important subclasses of ***C***. We feel that the most important subclass known so far is the class of regular clutters. For this class we may use Seymour's forbidden minor characterization to check whether or not a given clutter is a member. However, finding an efficient algorithm for doing this verification still is an unsolved problem.

Before we establish some properties of regular clutters, we recall two other results on totally c-optimal clutters. As we already have mentioned, Satyanarayana and Chang [8] established that multiterminal undirected network systems are totally c-optimal. In order to show this, they in fact proved the stronger result that such systems are totally amenable. Similarly, in Barlow [1] it is established that k-out-of-n systems are totally c-optimal, by showing that such systems are totally amenable. In this section we shall see that both these classes of clutters are regular. Thus, the result of Huseby [5] on the relation between regularity and total amenability generalizes both Satyanarayana and Chang [8] and Barlow [1].

It is very easy to see that k-out-of-n systems are regular. A detailed proof of this is given in Huseby [4]. This result can also be seen as a direct consequence of Theorem 3.1 as follows: It is well-known that the class of k-out-of-n systems is closed under minor operations. Thus, to show that k-out-of-n systems are regular, by the argument following Theorem 3.1, it is sufficient to show that none of the four types of clutters mentioned in Theorem 3.1, is k-out-of-n systems, which of course is trivial.

Similarly, one can prove that also multiterminal undirected network systems are regular, by noting that this class is closed under minor operations as well, and then verify that none of forbidden minors of Theorem 3.1 can be represented as multiterminal undirected network systems. However, the last verification is quite tedious, and provides little new insight. In this section we shall show that this result is a consequence of a much more general result.

It should be mentioned that the fact that also multiterminal undirected network systems are regular already is known from Huseby [4]. However, the proof given here is simpler. Moreover, in the light of the new proof an interesting parallel result can be obtained, yielding a new class of regular clutters associated with undirected networks.

In the development of the new results, we need the concept of rank functions. If $(F,\mathscr{M})$ is a matroid, for all $A \subseteq F$ we define the *rank function*, ρ, of the matroid as:

$$(4.1.) \qquad \rho(A) = \max\{|B| : B \subseteq A, M \not\subset B, \forall M \in \mathscr{M}\}$$

It is well-known that a matroid is uniquely determined by its rank function. Furthermore, we have the following fundamental result:

Proposition 4.1. Let ρ be an integer valued function defined for all subsets of a given set F. Then ρ is a rank function of a matroid if and only if the following three conditions hold:

(4.2.) $0 \leq \rho(A) \leq |A|$ for all $A \subseteq F$.

(4.3.) $A \subseteq B$ implies $\rho(A) \leq \rho(B)$ for all $A, B \subseteq F$.

(4.4.) $\rho(A \cup B) + \rho(A \cap B) \leq \rho(A) + \rho(B)$ for all $A, B \subseteq F$.

For more details and proofs we refer to Welsh [10].

In Huseby [5] the following characterization of regular clutters is given:

Proposition 4.2. A clutter $(E,\mathscr{P})$ with structure function ϕ is regular if and only if ϕ can be represented as:

(4.5.) $\phi(A) = 1 + \rho(A) - \rho(A \cup x)$ for all $A \subseteq E$.

where ρ is the rank function of some matroid $(E \cup x, \mathscr{M})$ and $x \notin E$ is the extension component.

It is also easy to see that we have the folowing converse relation:

Proposition 4.3. Let $(E \cup x, \mathscr{M})$ where $x \notin E$, be a matroid with rank function ρ, and let the function ϕ be defined by (4.5.). Then ϕ is binary and nondecreasing. In particular, ϕ is the structure function of the clutter $(E,(\mathscr{M}_x)_{+x})$.

It is well-known that the class of matroids is closed under restrictions. Furthermore, according to the definitions in this paper, it is also well-known that the contraction of a matroid $(F,\mathscr{M})$ w.r.t an element e is a matroid unless $\{e\} \in \mathscr{M}$. [In standard matroid theory it is common to modify the definition of matroid contraction such that the class of matroids is closed under such operations in general. See e.g. Welsh [10]. In order to avoid confusion we shall not do so here.]

Now, let $(E,\mathscr{P})$ be a regular clutter with corresponding matroid $(E \cup x, \mathscr{M})$. In the theory we are about to develop, the contraction and restriction of $(E \cup x, \mathscr{M})$ will play an essential role. Assuming that $(E,\mathscr{P})$ is nontrivial, both these minors will be matroids, and we shall refer to them as respectively the c-matroid and r-matroid corresponding to $(E,\mathscr{P})$. If ρ_{+x} and ρ_{-x} denote the rank functions of respectively the c-matroid and r-matroid, then by standard matroid theory, these rank functions are given by:

(4.6.) $\rho_{+x}(A) = \rho(A \cup x) - 1$, for all $A \subseteq E$.

(4.7.) $\rho_{-x}(A) = \rho(A)$, for all $A \subseteq E$.

Having introduced these concepts and results, we are now ready to state the main results of this section.

Theorem 4.4. Let $(E,\mathscr{P}^1), \dots, (E,\mathscr{P}^k)$ be regular nontrivial clutters with structure functions $\phi^1,\dots,\phi^k$ respectively. Introduce also $\phi = \min(\phi^1,\dots,\phi^k)$, and let $(E,\mathscr{P})$ denote the corresponding clutter. Assume that $(E,\mathscr{P}^1), \dots, (E,\mathscr{P}^k)$ all have the same corresponding r-matroid. Then $(E,\mathscr{P})$ is also regular having the same corresponding r-matroid as $(E,\mathscr{P}^1), \dots, (E,\mathscr{P}^k)$.

Proof: Let ρ^j denote the rank function of the matroid corresponding to $(E,\mathscr{P}^j)$, j=1,…,k. If x is the extension component, by Proposition 4.2, we have:

$$(4.8.)\qquad \phi^j(A) = 1 + \rho^j(A) - \rho^j(A\cup x),\ \text{ for all } A\subseteq E,\ j = 1,\ldots,k.$$

We then introduce:

$$(4.9.)\qquad \rho(B) = \max(\rho^1(B),\ldots,\rho^k(B)),\ \text{ for all } B\subseteq(E\cup x).$$

By (4.7.) the rank function of the r-matroid corresponding to $(E,\mathscr{P}^j)$, ρ^j_{-x}, is given by $\rho^j_{-x}(A) = \rho^j(A)$ for all $A\subseteq E$. Hence, since $(E,\mathscr{P}^1)$, … , $(E,\mathscr{P}^k)$ all have the same corresponding r-matroid, i.e. the ρ^j_{-x}-functions are equal, we must have:

$$(4.10.)\qquad \rho^1(A) = \cdots = \rho^k(A) = \rho(A),\ \text{ for all } A\subseteq E.$$

Inserting this into (4.8.) yields:

$$(4.11.)\qquad \phi^j(A) = 1 + \rho(A) - \rho^j(A\cup x),\ \text{ for all } A\subseteq E,\ j = 1,\ldots,k.$$

Hence, the structure function ϕ is given by:

$$(4.12.)\qquad \phi(A) = \min(\phi^1(A),\ldots,\phi^k(A)) = 1 + \rho(A) - \max(\rho^1(A\cup x),\ldots,\rho^k(A\cup x))$$

$$= 1 + \rho(A) - \rho(A\cup x),\ \text{ for all } A\subseteq E.$$

Since obviously ϕ is nondecreasing, $\phi(A) = 1$ implies $\phi(B) = 1$, whenever $A\subseteq B$, and $\phi(A) = 0$ implies $\phi(B) = 0$, whenever $B\subseteq A$. Thus, since by (4.12.) $\phi(A) = 1$ if and only if $\rho(A) = \rho(A\cup x)$, and $\phi(A) = 0$ if and only if $\rho(A) = \rho(A\cup x)-1$,we get that ρ must satisfy:

$$(4.13.)\qquad \rho(A) = \rho(A\cup x) \text{ implies } \rho(B) = \rho(B\cup x),\ \text{ for all } A\subseteq B\subseteq E.$$

$$\rho(A) = \rho(A\cup x)-1 \text{ implies } \rho(B) = \rho(B\cup x)-1,\ \text{ for all } B\subseteq A\subseteq E.$$

We now show that ρ is a rank function of a matroid. To do so, we must verify that ρ satisfies (4.2.), (4.3.) and (4.4.) (where of course F is replaced by $(E\cup x)$).

It is very easy to see that ρ satisfies (4.2.) and (4.3.). This follows more or less directly since these conditions are satisfied by all the ρ^j-s, and we skip further details here. Thus, we turn to the property (4.4.), and let A, $B\subseteq(E\cup x)$.

If $x\notin(A\cap B)$, then by (4.10.) $\rho(A\cap B) = \rho^j(A\cap B)$ for j = 1,…,k. Moreover, we may of course always choose $j\in\{1,\ldots,k\}$ such that $\rho^j(A\cup B) = \max(\rho^1(A\cup B),\ldots,\rho^k(A\cup B)) = \rho(A\cup B)$. Thus, since (4.4.) is satisfied by ρ^j, we get:

$$(4.14.)\qquad \rho(A\cup B) + \rho(A\cap B) = \rho^j(A\cup B) + \rho^j(A\cap B) \le \rho^j(A) + \rho^j(B) \le \rho(A) + \rho(B),$$

so (4.4.) holds true.

Assume now that $x\in(A\cap B)$. We then consider two possible cases:

Case 1. $\rho(A) = \rho(A\backslash x)$.

By (4.13.) this implies that $\rho(A\cup B) = \rho((A\cup B)\backslash x)$. Moreover, by (4.10.) $\rho((A\cup B)\backslash x) = \rho^j((A\cup B)\backslash x)$ for $j = 1,\dots,k$. By the definition of ρ we have that $\rho^j(A\cup B) \leq \rho(A\cup B)$ for $j = 1,\dots,k$. On the other hand, since ρ^j satisfies (4.3.), $\rho^j((A\cup B)\backslash x) \leq \rho^j(A\cup B)$, for $j = 1,\dots,k$. Combining all this we get that $\rho^j(A\cup B) = \rho(A\cup B)$ for $j = 1,\dots,k$. Moreover, we may choose $j\in \{1,\dots,k\}$ such that $\rho^j(A\cap B) = \max(\rho^1(A\cap B),\dots,\rho^k(A\cap B)) = \rho(A\cap B)$. Thus, since (4.4.) is satisfied by ρ^j, as above we get that (4.14.) holds true again. Hence, (4.4.) is satisfied in this case as well.

Case 2. $\rho(A) = \rho(A\backslash x) + 1$.

By (4.13.) this implies that $\rho(A\cap B) = \rho((A\cap B)\backslash x)+1$. Moreover, by (4.10.) $\rho((A\cap B)\backslash x) = \rho^j((A\cap B)\backslash x)$ for $j = 1,\dots,k$. Combining this and noting that $((A\cap B)\backslash x) = ((A\backslash x)\cap B)$, we get that $\rho(A\cap B) = \rho^j((A\backslash x)\cap B)+1$ for $j = 1, \dots , k$.

We then choose once again $j \in \{1, \dots , k\}$ such that $\rho^j(A\cup B) = \max(\rho^1(A\cup B), \dots , \rho^k(A\cup B)) = \rho(A\cup B)$. Since $x\in A\cap B$, we may write $(A\cup B) = ((A\backslash x)\cup B)$. Hence, since (4.4.) is satisfied by ρ^j, by the definition of ρ, and the assumption that $\rho(A) = \rho(A\backslash x) + 1$, it follows that:

$$\text{(4.15.)}\qquad \rho(A\cup B) + \rho(A\cap B) = \rho^j((A\backslash x)\cup B) + \rho^j((A\backslash x)\cap B) + 1$$

$$\leq \rho^j(A\backslash x) + \rho^j(B) + 1 \leq \rho(A\backslash x) + 1 + \rho(B) = \rho(A) + \rho(B)$$

so (4.4.) holds true again.

Thus, we have proved that ρ is the rank function of a matroid. Hence, by Proposition 4.2. $(E,\mathscr{P})$ is regular, and by (4.10.) and (4.7.) it has the same corresponding r-matroid as $(E,\mathscr{P}^1),\dots,(E,\mathscr{P}^k)$.■

Theorem 4.5. Let $(E,\mathscr{P}^1), \dots , (E,\mathscr{P}^k)$ be regular nontrivial clutters with structure functions $\phi^1,\dots,\phi^k$ respectively. Introduce also $\phi = \max(\phi^1,\dots,\phi^k)$, and let $(E,\mathscr{P})$ denote the corresponding clutter. Assume that $(E,\mathscr{P}^1), \dots , (E,\mathscr{P}^k)$ all have the same corresponding c-matroid. Then $(E,\mathscr{P})$ is also regular having the same corresponding c-matroid as $(E,\mathscr{P}^1), \dots , (E,\mathscr{P}^k)$.

Proof: As above we let ρ^j denote the rank function of the matroid corresponding to $(E,\mathscr{P}^j)$, $j=1,\dots,k$. If x is the extension component, then ϕ^j is given by (4.8.).

We then introduce:

$$\text{(4.16.)}\qquad \rho(B) = \min(\rho^1(B),\dots,\rho^k(B)), \quad \text{for all } B\subseteq(E\cup x)$$

By (4.6.) the rank function of the c-matroid corresponding to $(E,\mathscr{P}^j)$, ρ^j_{+x}, is given by $\rho^j_{+x}(A) = \rho^j(A\cup x)-1$ for all $A\subseteq E$. Hence, since $(E,\mathscr{P}^1), \dots , (E,\mathscr{P}^k)$ all have the same corresponding c-matroid, i.e. the ρ^j_{+x}-functions are equal, we must have:

$$\text{(4.17.)}\qquad \rho^1(A\cup x) = \cdots = \rho^k(A\cup x) = \rho(A\cup x), \text{ for all } A\subseteq E.$$

Inserting this into (4.8.) yields:

$$(4.18.)\qquad \phi^j(A) = 1 + \rho^j(A) - \rho(A\cup x), \text{ for all } A\subseteq E,\ j = 1,\ldots,k.$$

Hence, the structure function ϕ is given by:

$$(4.19.)\qquad \phi(A) = \max(\phi^1(A),\ldots,\phi^k(A)) = 1 + \max(\rho^1(A),\ldots,\rho^k(A)) - \rho(A\cup x)$$

$$= 1 + \rho(A) - \rho(A\cup x), \quad \text{for all } A\subseteq E.$$

Since obviously ϕ is nondecreasing, (4.13.) is valid in this case as well.

We now show that ρ is a rank function of a matroid. As in the proof of Theorem 4.4 we restrict ourselves to proving that ρ satisfies (4.4.), and let $A, B\subseteq(E\cup x)$.

If $x\in(A\cup B)$, then by (4.17.) $\rho(A\cup B) = \rho^j(A\cup B)$ for $j = 1,\ldots,k$. Moreover, we may choose $j\in\{1, \ldots, k\}$ such that $\rho^j(A\cap B) = \max(\rho^1(A\cap B), \ldots, \rho^k(A\cap B)) = \rho(A\cap B)$. Thus, since (4.4.) is satisfied by ρ^j, as in the proof of Theorem 4.4, we get that (4.14.) holds true, implying that (4.4.) is satisfied.

Assume now that $x\notin(A\cup B)$. We then consider two possible cases:

Case 1. $\rho(A) = \rho(A\cup x)$.

By (4.13.) and (4.17.) this implies that $\rho(A\cup B) = \rho(A\cup B\cup x) = \rho^j(A\cup B\cup x)$ for $j = 1, \ldots, k$. We then choose $j\in\{1, \ldots, k\}$ such that $\rho^j(A\cap B) = \max(\rho^1(A\cap B),\ldots,\rho^k(A\cap B)) = \rho(A\cap B)$. Hence, since (4.4.) is satisfied by ρ^j, by the definition of ρ, and the assumption that $\rho(A) = \rho(A\cup x)$, it follows that:

$$(4.20.)\qquad \rho(A\cup B) + \rho(A\cap B) = \rho^j(A\cup B\cup x) + \rho^j(A\cap B)$$

$$\leq \rho^j(A\cup x) + \rho^j(B) \leq \rho(A\cup x) + \rho(B) = \rho(A) + \rho(B)$$

Hence, (4.4.) holds true.

Case 2. $\rho(A) = \rho(A\cup x) - 1$.

By (4.13.) this implies that $\rho(A\cap B) = \rho((A\cap B)\cup x)-1$. Moreover, (4.17.) $\rho((A\cap B)\cup x) = \rho^j((A\cap B)\cup x)$ for $j = 1,\ldots,k$. Since $\rho^1,\ldots,\rho^k$ satisfy (4.2.) and (4.4.), $\rho^j((A\cap B)\cup x) \leq \rho^j(A\cap B) + \rho^j(x) \leq \rho^j(A\cap B) + 1$ for $j = 1,\ldots,k$. Combining all this we get that $\rho(A\cap B) \leq \rho^j(A\cap B)$ for $j = 1,\ldots,k$. On the other hand, by the definition of ρ we have that $\rho^j(A\cap B) \leq \rho(A\cap B)$ for $j = 1,\ldots,k$. Thus, $\rho(A\cap B) = \rho^j(A\cap B)$ for $j = 1,\ldots,k$.

We then choose $j\in\{1,\ldots,k\}$ such that $\rho^j(A\cup B) = \max(\rho^1(A\cup B),\ldots,\rho^k(A\cup B)) = \rho(A\cup B)$. Hence, since (4.4.) is satisfied by ρ^j, as usually, we get that (4.14.) holds true, implying that (4.4.) is satisfied in this case as well.

Thus, we have proved that ρ is the rank function of a matroid. Hence, by Proposition 4.2. $(E,\mathscr{P})$ is regular, and by (4.17.) and (4.6.) it has the same corresponding c-matroid as $(E,\mathscr{P}^1)$, … , $(E,\mathscr{P}^k)$. ■

Having proved these fundamental properties of regular clutters, we now turn to some applications of these results to undirected network systems.

We observe that both Theorem 4.4 and Theorem 4.5 start out with a set of clutters already known to be regular, and then show that a certain combination of these clutters also is regular. Hence, in order to use these theorems, we must have some regular clutters to start with. In the applications we are about to present, this will be 2-terminal undirected networks.

To see that 2-terminal undirected networks are regular, we use the well-known fact that if F is the set of edges in an undirected network, and $\mathscr{M}$ is the family of minimal circuits in the network, then $(F,\mathscr{M})$ is a matroid, called the *circuit matroid* of the network. (See e.g. Welsh [10].) Let G be an undirected network with edge set E and two terminals, S and T, and let the clutter $(E,\mathscr{P})$ correspond to the system which is functioning if S and T can communicate through G, (i.e. $\mathscr{P}$ is the family of minimal paths from S to T in the graph). Then $(E,\mathscr{P})$ is regular with corresponding matroid $(E \cup x,\mathscr{M})$ where $\mathscr{M}$ is the family of minimal circuits in the network G' obtained from G by adding the edge x between S and T. This follows since by the above result $(E \cup x,\mathscr{M})$ is a matroid, and clearly $(E, \mathscr{P}) = (E,(\mathscr{M}_x)_{+x})$.

An illustration of a 2-terminal undirected network system is provided in Figure 4.1, while in Figure 4.2 we have shown the system with the edge, *x*, added between the terminals S and T.

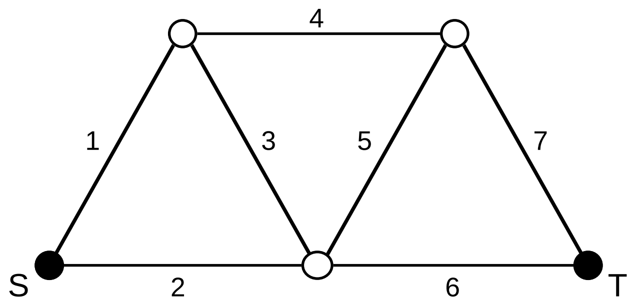

Figure 4.1. A 2-terminal undirected network system.

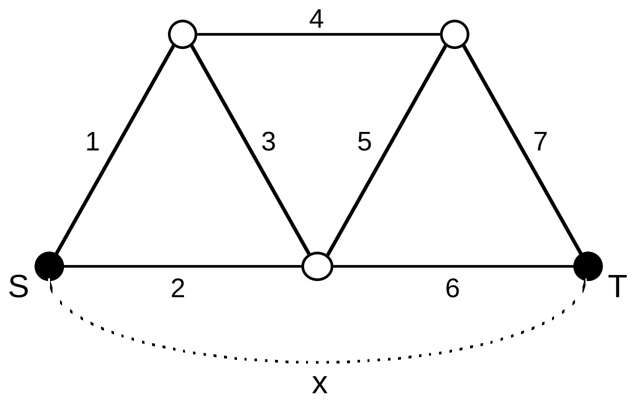

Figure 4.2. The edge, *x,* is added between the terminals, S and T.

We now focus on the r-matroid and c-matroid corresponding to a 2-terminal undirected network system. It is easily seen that if $(E,\mathscr{P})$ is such a system, and $(E,\mathscr{P}) \leftrightarrow (E\cup x,\mathscr{M})$, then $\mathscr{M}_{-x}$ is the family of minimal circuits in the original network, i.e. the network without the edge x added. Furthermore, $\mathscr{M}_{+x}$ is the family of minimal circuits in the network obtained from the original network by adjoining the two terminals. In Figure 4.3 we have illustrated the network with family of minimal circuits corresponding to the c-matroid of the network shown in Figure 4.1.

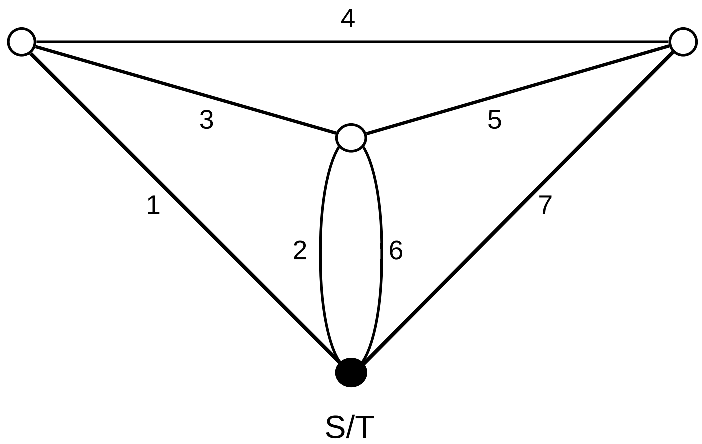

Figure 4.3. The network obtained by adjoining the terminals S and T.

Now in order to use Theorem 4.4 and Theorem 4.5, we search for different clutters having either equal r-matroids or equal c-matroids. The following theorem deals with the case of 2-terminal undirected network systems having equal r-matroids.

Theorem 4.6. Let G be an undirected network with edge set E, let $\{S_1,T_1\},\ldots,\{S_k,T_k\}$ be k pairs of nodes (not necessarily disjoint) in G, and let $(E,\mathscr{P}^j)$ be the clutter with structure function, ϕ^j, corresponding to the system which is functioning if S_j and T_j can communicate through G, $j = 1,\ldots,k$. Furthermore, let $(E,\mathscr{P})$ be the clutter corresponding to the structure function $\phi = \min(\phi^1,\ldots,\phi^k)$, (i.e. $(E,\mathscr{P})$ corresponds to a system functioning if S_j and T_j can communicate through G for all $j = 1,\ldots,k$). Then $(E,\mathscr{P})$ is regular, and the r-matroid corresponding to $(E,\mathscr{P})$ is the circuit matroid of the network.

Proof: Since $(E,\mathscr{P}^1),\ldots,(E,\mathscr{P}^k)$ are clutters corresponding to 2-terminal undirected network systems on the same underlying network G, these clutters are regular with the same r-matroid, namely the circuit matroid of G. Hence, the result follows directly from Theorem 4.4. ■

As a simple corollary of this result, we now obtain that all multiterminal undirected network systems are regular:

Corollary 4.7. Let G be an undirected network with edge set E and and a set of terminals, T_0, $T_1,\ldots,T_k$, and let $(E,\mathscr{P})$ be the clutter corresponding to the system which is functioning if all the (k+1) terminals can communicate through G. Then $(E,\mathscr{P})$ is regular.

Proof: Let $(E,\mathscr{P}^j)$ be the clutter corresponding to the system which is functioning if T_{j-1} and T_j can communicate through G, and let ϕ^j be the structure function of $(E,\mathscr{P}^j)$ $j = 1,\ldots,k$. Then it is easily seen that the structure function of $(E,\mathscr{P})$, ϕ, is given by $\phi = \min(\phi^1,\ldots,\phi^k)$. Thus, the result follows by Theorem 4.6. ■

We observe that the class of clutters corresponding to multiterminal undirected network systems is actually a subclass of the more general class of regular clutters described in Theorem 4.6. Thus, Theorem 4.6 yields a true generalization of the results of Satyanarayana and Chang [8].

Now, we turn to combinations of 2-terminal undirected network systems with equal c-matroids, and consider once again the system illustrated in Figure 4.1 and with the circuit matroid of the network shown in Figure 4.3 as c-matroid. Clearly there exist lots of other systems having the same c-matroid. One such system is the 2-terminal undirected network system shown in Figure 4.4.

Let $(E,\mathscr{P}^1)$ and $(E,\mathscr{P}^2)$ denote the clutters corresponding to the systems in Figure 4.1 and Figure 4.4 respectively. We introduce also the structure function, ϕ^j, of $(E,\mathscr{P}^j)$, $j = 1, 2$, and let $(E,\mathscr{P})$ denote the clutter with structure function $\phi = \max(\phi^1, \phi^2)$. Using Theorem 4.5 it is of course evident that $(E,\mathscr{P})$ is regular. However, since $(E,\mathscr{P}^1)$ and $(E,\mathscr{P}^2)$ correspond to 2-terminal systems on different networks, it is difficult to get a practical interpretation of the system corresponding to $(E,\mathscr{P})$.

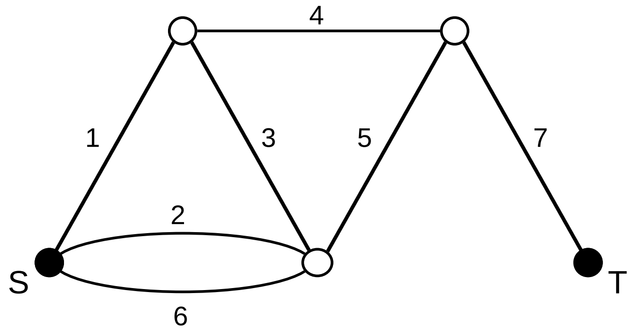

Figure 4.4. Another 2-terminal undirected network system with the same c-matroid.

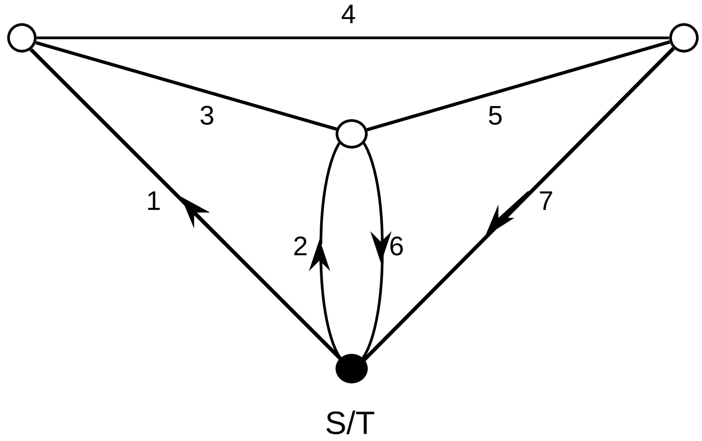

Figure 4.5. A 1-terminal representation of (E, $\mathscr{P}^1$).

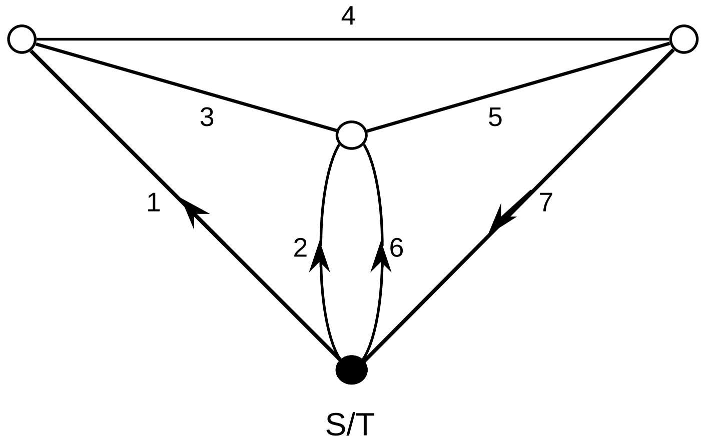

Figure 4.5. A 1-terminal representation of (E,$\mathscr{P}^1$).

There exists however another way of representing the clutters (E,$\mathscr{P}^1$) and (E,$\mathscr{P}^2$) based on the network shown in Figure 4.3, yielding a more intuitive interpretation of (E,$\mathscr{P}$). Thus, in Figure 4.5 and Figure 4.6 we have represented respectively (E,$\mathscr{P}^1$) and (E,$\mathscr{P}^2$) as 1-terminal network systems, where the edges adjacent to the terminal are directed. A system is functioning if the terminal is contained in at least one functioning circuit, with excactly one edge directed into the terminal and one edge directed out from the terminal.

We observe that the only difference between the two networks is the direction of the edge 6. Since the system corresponding to (E,$\mathscr{P}$) is functioning if and only if at least one of the two 1-terminal systems is functioning, a network representation of (E,$\mathscr{P}$) is obtained by replacing the edge 6 by an undirected edge, as shown in Figure 4.7.

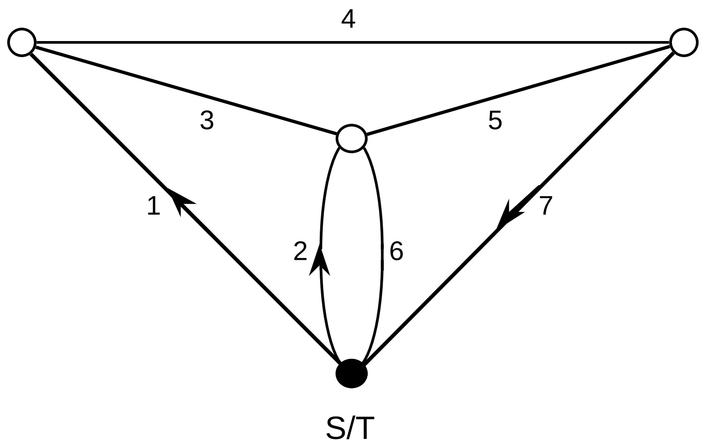

Figure 4.7. A 1-terminal representation of $(E,\mathscr{P})$.

By combining, as above, different 1-terminal systems on the same underlying network and with the same terminal node, new 1-terminal systems with more or less directed edges coming in and out of the terminal may be constructed. Especially, let $(E,\mathscr{P}^1),\ldots,(E,\mathscr{P}^k)$ be the family of *all* clutters representable as 1-terminal systems on the same underlying network, with the same terminal node and with only directed edges adjacent to the terminal. Furthermore, let $\phi^1,\ldots,\phi^k$ be the structure functions of $(E,\mathscr{P}^1),\ldots,(E,\mathscr{P}^k)$ respectively, and let $(E,\mathscr{P})$ be the clutter with structure function $\phi = \max(\phi^1,\ldots,\phi^k)$. Then it is easy to see that $(E,\mathscr{P})$ can be represented as a 1-terminal system on the same underlying network, with the same terminal node and with only undirected edges. We shall call such a system a *1-terminal undirected circulation system*. By Theorem 4.5 and the above discussion the following result is imediate:

Theorem 4.8. A 1-terminal undirected circulation system on a network, G, is regular. Moreover, the c-matroid of the system is equal to the circuit matroid of G.

Furthermore, using Theorem 4.5. once more, the following multiterminal version of result can be proved:

Theorem 4.9. Let $(E,\mathscr{P}^1),\ldots,(E,\mathscr{P}^k)$ be 1-terminal undirected circulation systems on the same network, G. Furthermore, let $\phi^1,\ldots,\phi^k$ be the structure functions of $(E,\mathscr{P}^1),\ldots,(E,\mathscr{P}^k)$ respectively, and let $(E,\mathscr{P})$ be the clutter with structure function $\phi = \max(\phi^1,\ldots,\phi^k)$. Then $(E,\mathscr{P})$ is regular, and the c-matroid of the system is equal to the circuit matroid of G.

Considering the clutters $(E,\mathscr{P}^1),\ldots,(E,\mathscr{P}^k)$ and $(E,\mathscr{P})$ introduced in Theorem 4.9, we observe that if $T_1,\ldots,T_k$ are the set of terminals of $(E,\mathscr{P}^1),\ldots,(E,\mathscr{P}^k)$, then the clutter $(E,\mathscr{P})$ corresponds to a system which is functioning if and only if at least one of the terminals $T_1,\ldots,T_k$ is contained in a functioning circuit. Thus, it is reasonable to call a system such as $(E,\mathscr{P})$, a *multiterminal undirected circulation system*.

5. Final remarks.

In the first part paper we have studied the relation between the concepts of regularity, total amenability and total c-optimality. In the light of this study it is clear that regularity is not a necessary condition for total amenability, and total amenability is not implied by total c-optimality. Due to Seymour [9], the class of regular clutters is characterized in a nice way. However, for the classes of totally amenable clutters and totally c-optimal clutters, there is still no such characterization available.

In the second part of this paper we have focused on properties of the class of regular clutters. Especially, we have found some new interesting subclasses of this class. Over the last decades reliability computations of multiterminal undirected network systems have been studied in a lot of papers, and many efficient reduction methods have been developed. It turns out that for many of these results there exist nice parallel results on multiterminal undirected circulation systems. We will return to this subject in a forthcoming paper.

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